Backpaper - Differential Equations (2018-19)

Time: 3 hours. *Attempt all questions.*

1. Find the general solution of $u'' + 2u' + u = e^{-x}$. **[5 marks]**

2. Find the general solution of $4xu'' + 2u' + u = 0$. **[5 marks]**

3. Find the general solution of

$$\begin{cases} \frac{dx}{dt} &= x + y \\ \frac{dy}{dt} &= 4x - 2y \end{cases} \quad \textbf{[5 marks]}$$

4. Solve for $u = u(x, y)$ in the PDE $(1 + x^2)u_x + u_y = 0$. **[5 marks]**

5. Solve the heat equation $u_t = ku_{xx}$ for $-l \le x \le l,\ t > 0$, with periodic boundary conditions [i.e. $u(-l, t) = u(l, t)$ and $u_x(-l, t) = u_x(l, t)$], and initial profile $u(0, x) = \phi(x)$, $-l \le x \le l$. Assume that ϕ is a C^2 (twice continuously differentiable) function. **[7 marks]**

6. Solve $u_{tt} = c^2 u_{xx}$ with $u(x, 0) = e^x$, $u_t(x, 0) = \sin x$. **[5 marks]**

7. Let $u = u(x, y)$ be a harmonic function in the disk $D = \{r < 2\}$ with $u = 2\sin 2\theta + 1$ for $r = 2$.

   (a) Find the maximum value of u in $\bar{D}$. **[2 marks]**

   (b) Calculate the value of u at the origin. **[3 marks]**

8. Define the Green's function for the operator $-\Delta$ in a domain D at the point $x_0 \in D$. **[5 marks]**

9. Let $F : I \to \mathbf{R}^d$ be a locally Lipschitz function of at most linear growth, that is $|F(u)| \le C(1 + |u|)$ for some constant C. Show that for each $u_0 \in \mathbf{R}^d$ and $t_0 \in \mathbf{R}$ there exists a unique *global* solution $u : \mathbf{R} \to \mathbf{R}^d$ to the Cauchy problem $\partial_t u(t) = F(u(t)),\ t \in \mathbf{R};\ u(t_0) = u_0$. **[8 marks]**